HINTIKKA'S THEOREM DOES NOT HOLD IN NON-AXIOMATIC LOGIC

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Abstract: *Hintikka's theorem relates what is impossible to what is forbidden. It provides that if something is impossible, that cannot be permitted. There are logical demonstrations of the theorem. Those demonstrations follow requirements of classical, modal, and deontic logics. However, there are also accounts based on psychological theories trying to explain why people's tendency should be to reject it. I will attempt to account for the probable rejection of the theorem by people too. But my explanation will resort to Non-Axiomatic Logic. I will argue that, from the latter logic, linking possibility and prohibition is preferable to linking impossibility and prohibition. So, Hintikka's theorem does not hold in Non-Axiomatic Logic.*

Keywords: *Hintikka's theorem, impossibility, Non-Axomatic Logic, possibility, prohibition.*

Introduction

Hintikka's theorem is well-known. It provides that if something cannot be the case, that is forbidden. It is often expressed as follows:

(1) $\forall x (\neg\Diamond x \Rightarrow \neg Px)$

Other ways to express the theorem are to be found in the literature (see, e.g., (12) in Øhrstrøm, Zeller, & Sandborg-Petersen, 2012, or (HT) in López-Astorga, 2017). (1) is a formula in first-order predicate calculus. '$\forall$' represents the universal quantifier, '$\neg$' is the negation symbol, '$\Diamond$' stands for the modal operator of possibility, '$\Rightarrow$' denotes the material conditional, and 'P' symbolizes the deontic operator of permission.

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The theorem is counterintuitive and accordingly hard to accept. One might ask "…why should what is impossible also be forbidden? What is the point in not permitting the impossible?" (Øhrstrøm et al., 2012, p. 451). We have logic demonstrations of it (see, e.g., Prior, 2012, and the analysis of the latter paper in Øhrstrøm et al., 2012). Those demonstrations respect the technical meanings of 'possibility' in modal logic and 'permission' in deontic logic. But we can also find works trying to explain the reasons why individuals' general tendency should be not to admit Hintikka's theorem. To do that, for example, a contemporary cognitive theory was considered. That theory is the theory of mental models (e.g., Johnson-Laird, 2023; Jonson-Laird, Byrne, & Khemlani, 2024). Based on this theory, people build mental representations when processing sentences. Given a sentence such as (1) expressed in natural language, those mental representations can prevent from accepting Hintikka's theorem (López-Astorga, 2017).

My purpose here is to attempt to show that in Non-Axiomatic Logic (e.g., Wang, 2013, 2023. From now on, I will use 'NAL' to refer to the latter logic; it is the usual abbreviation to name it) the theorem does not hold. NAL is the logic from which NARS (Non-Axiomatic Reasoning System; see also, e.g., Wang, 2006), that is, a computer program, comes. NARS is not intended to work as the human mind, but it does try to make inferences in a similar manner to people (e.g., Wang, 2013). I will not review whether NARS makes inferences in that way. I will only propose that its logical system, that is, NAL, does not allow accepting (1). My point will be just that, in this case, NAL does appear to work in a way akin to our mind.

The present paper will be divided into two sections. In the first one, I will describe the components NAL seems to need to deal with sentences such as (1). In the second section, I will present my account of the reasons why sentences such as (1) should be rejected in NAL.

A brief description of NAL

The statements in NAL are 'inheritance statements' linking subjects and predicates (e.g., Wang, 2013). A typical inheritance statement in NAL is (2).

(2) "$S \rightarrow P \langle f, c \rangle$" (Wang, 2013, p. 40; Definition 3.8).

In (2), S denotes the subject of the inheritance statement. Being the subject means being in a set: in the extension of the predicate, which is P in (2). In turn, P is also an element in a set: the intension of S. Thus, what copula '$\rightarrow$' in (2) provides is "…that S is in the extension of P and P is in the intension of S" (Wang, 2013, p. 40; Definition 3.8; italics in text). This is important because, as indicated in most works explaining NAL, those are not the habitual meanings for 'extension' and 'intension' in logic. While '$\rightarrow$' has isomorphic properties with '$\Rightarrow$' (e.g., Wang, 2013; Definitions 9.2 and 9.3), what (2) establishes is what is expressed in (3).

(3) "$(S \rightarrow P) \Leftrightarrow (S^E \subseteq P^E) \Leftrightarrow (P^I \subseteq S^I)$" (Wang, 2013, p. 20; Theorem 2.4).

In (3), '$\Leftrightarrow$' represents biconditional relation as understood in first-order predicate calculus, X^E stands for the extension of X, and X^I denotes the intension of X.

Regarding $\langle f, c\rangle$, it is the truth value of the statement. The first component, f, is 'frequency'. It is calculated by means of the formulae in (4).

(4) "$f = w^+/w$" (Wang, 2013, p. 29; Definition 3.3); "$w^+ = |S^E \cap P^E|+|P^I \cap S^I|$" (Wang, 2013, p. 28; Definition 3.2); "$w = |S^E|+|P^I|$" (Wang, 2013, p. 28; Definition 3.2).

As it can be inferred from (4), w^+ refers to the 'positive evidence' of the statement, and w stands for the 'total evidence' of that very statement.

As far as c in (2) is concerned, it is the 'confidence' of the statement. NAL also has a formula to calculate it:

(5) "$c = w/(w + k)$" (Wang, 2013, p. 29; Definition 3.3).

The role of k in (5) is that of a constant. In NAL, it is habitual to consider it to be equal to 1 (for reasons for that, see, e.g., Wang, 2013).

Components f and c are important in NAL in several senses. For the present paper, one of the reasons why they are relevant is that one might think that f and c play the role of quantifiers in other logics. NAL works with a basic assumption: the Assumption of Insufficient Knowledge and Resources

(AIKR; in addition to Wang, 2013, this assumption is addressed in detail in, e.g., Wang, 2011). The assumption implies that there are always doubts about the evidence reviewed. It is always possible to get new evidence, which can change the current values of *f* and *c*. From this point of view, we can think that if we use *f* and *c*, quantifiers such as the existential and the universal quantifiers in first-order predicate calculus become irrelevant (e.g., Wang, 2023).

On the other hand, there are many inference rules in NAL. The system enables to make inferences such as deductions, inductions, abductions, revisions, etc. (e.g., Wang, 2013; for a brief explanation of some of the rules, see, in addition, Wang, 2023). However, the rule that is interesting here is the 'choice rule'. Given a question such as '? → *P*', that is, a question about the most appropriate subject for a predicate, NAL also has a formula to determine what option to choose. That formula allows calculating *e*, that is, the 'expectation value'. It is the formula in (6).

(6) "$e = (w^+ + k/2)/(w + k)$", or "$e = c \times (f - ½) + ½$" (Wang, 2013, p. 48; Table 4.2).

The alternative with highest *e* will be the alternative to select.

All this can also be shown by means of an example. Taking AIKR into account, let us suppose a fictional scenario such as the following.

The system knows ten people. eight of those people are Asian, and two of them are European. Out of the eight Asian people, five are Chinese and three are Japanese. One European person is German, and the other one is Portuguese. This information enables to build inheritance statements (7) to (16).

(7) *Asian* → *Person* (1, 0.89)

This is because $w = 8$ and $w^+ = 8$ for (7).

(8) *European* → *Person* (1, 0.67)

This is because $w = 2$ and $w^+ = 2$ for (8).

(9) *Chinese* → *Person* (1, 0.83)

This is because $w = 5$ and $w^+ = 5$ for (9).

(10) *Japanese* → *Person* (1, 0.75)

This is because $w = 3$ and $w^+ = 3$ for (10).

(11) *German* → *Person* (1, 0.5)

This is because $w = 1$ and $w^+ = 1$ for (11).

(12) *Portuguese* → *Person* (1, 0.5)

This is because $w = 1$ and $w^+ = 1$ for (12).

(13) *Chinese* → *Asian* (1, 0.86)

This is because $w = 6$ and $w^+ = 6$ for (13) (if $Person^I$ is the intension of *Person*, $\{Chinese\} \in Person^I$, and $\{Asian\} \in Person^I$).

(14) *Japanese* → *Asian* (1, 0.8)

This is because $w = 4$ and $w^+ = 4$ for (14) ($\{Japanese\} \in Person^I$, and $\{Asian\} \in Person^I$).

(15) *German* → *European* (1, 0.67)

This is because $w = 2$ and $w^+ = 2$ for (15) ($\{German\} \in Person^I$, and $\{European\} \in Person^I$).

(16) *Portuguese* → *European* (1, 0.67)

This is because $w = 2$ and $w^+ = 2$ for (16) ($\{Portuguese\} \in Person^I$, and $\{European\} \in Person^I$).

With these data, NAL can respond to questions such as '? → *Person*', '? → *Asian*', or '? → *European*'. In the case of the first question, that is, '? → *Person*', we need to calculate *e* for inheritance statements (7) to (12). Let *e*(7), *e*(8), *e*(9), *e*(10), *e*(11), and *e*(12) be the expectation values of, respectively, (7), (8), (9), (10), (11), and (12). (6) allows calculating them.

-*e*(7) = 0.94
-*e*(8) = 0.83
-*e*(9) = 0.92
-*e*(10) = 0.88
-*e*(11) = 0.75
-*e*(12) = 0.75

Because the highest value is *e*(7), the answer to '? → *Person*' would be *Asian*.

If the question were '? → *Asian*', we would require the values of *e* for (13) and (14). Let *e*(13) and *e*(14) be the expectation values of, respectively, (13) and (14). Then,

-*e*(13) = 0.93
-*e*(14) = 0.9

Since *e*(13) > *e*(14), the response would be *Chinese* in this case.

Finally, the values of *e* necessary to respond to '? → *European*' would be those of (15) and (16). Let *e*(15) and *e*(16) be the expectation values of, respectively, (15) and (16). (6) leads us to:

-*e*(15) = 0.83
-*e*(16) = 0.83

In this situation, the system could not choose between *German* and *Portuguese*, as the expectation value is the same for both (15) and (16). Beyond the way NAL can solve difficulties such as this one, what is important now is that the components of this logic described above can show that statements akin to (1) would not be prioritized in it. The next section addresses this point.

Hintikka's theorem and NAL

To consider (1) from NAL, the first thing to do is to translate a formula such as (1), which is a formula in first-order predicate calculus including operators from modal and deontic logics, into an inheritance statement such as those of NAL. The universal quantifier is not a problem. As said, if truth values such as *f* and *c* are included, no quantifier should be used. We are never sure about evidence in NAL. So, we cannot state definitively, for example, that all elements in a set are a subset of another set, or that an intersection between two sets exists. In NAL, the values obtained with its formulae are always variable. Thus, (1) can be transformed into (17).

(17) $\neg\Diamond x \Rightarrow \neg Px \langle f_x, c_x \rangle$

This does not suffice. The material conditional is only used in NAL at the meta-level to describe it (e.g., Wang, 2013). Hence, '⇒' needs to be replaced by '→'. As indicated, there is an isomorphism between the material conditional in classical logic and the inheritance copula in NAL (e.g., Wang, 2013; Definitions 9.2 and 9.3). Besides, transformations of conditionals in classical logic into inheritance statements in NAL are to be found in the literature. For example, there are works in which that was done to apply NAL to philosophical frameworks (see, e.g., López-Astorga, 2024, where NAL is combined with the testability process Carnap, 1936, 1937, proposed). So, one might think that changing (17) for (18) is justified.

(18) $\neg\Diamond x \rightarrow \neg Px \langle f_x, c_x \rangle$

The problems remaining are those caused by modal operator '◊' and deontic operator 'P'. NAL can remove those problems in several ways. Following works such as Wang (2013), one of these ways is to deem them as terms with extension and intension. '◊' can refer to *Possible*, and 'P' can denote *Permitted*. Given that in (18) both terms are negated, we should think about terms such as *Impossible* and *Forbidden*. That allows us to come to (19).

(19) $\textit{Impossible} \rightarrow \textit{Forbidden} \langle f_x, c_x \rangle$

At this point, to know how (19) would be processed in NAL, we would have to calculate f_x and c_x. That does not seem to be easy. However, there are other options that may not be so difficult. For instance, we can think about the status of (19) in NAL considering at the same time both an inheritance statement such as (20)

(20) $Possible \rightarrow Forbidden \langle f_y, c_y \rangle$

And a question such as (21).

(21) $? \rightarrow Forbidden$

According to (6), the answer to (21) could be neither *Possible*, by virtue of (20), nor *Impossible*, by virtue of (19). It could be another term different from both the system knows. But we can argue that NAL will always prefer *Possible*, or (20), over *Impossible*, or (19), in this case. If this is shown, we will be able to claim that the inheritance statements similar to what (1) expresses have low frequency values in NAL.

As indicated above, "…why should what is impossible also be forbidden? What is the point in not permitting the impossible?" (Øhrstrøm et al., 2012, p. 451) are valid questions. Questions such as these ones make sense because in real life we hardly find impossible actions that are forbidden. I am not saying that we cannot find impossible and forbidden actions. What I am saying is that it is difficult to find them.

The opposite happens in the case of (20). Most forbidden conducts are possible conducts. Therefore, we can think that NAL always has evidence in favor of (20), no matter how little information it has. Let w_{19}^+ and w_{20}^+ be the positive evidence in favor of, respectively, (19) and (20). It is obvious that (22) holds.

(22) $w_{20}^+ > 0$

But (23) is not obvious.

(23) $w_{19}^+ > 0$

What does be also evident is that $w_{19}^+ < w_{20}^+$.

Let w_{19} and w_{20} be the total evidence for, respectively, (19) and (20). In a fictional scenario in which the amount of evidence for (19) and (20) is the same, that is, in a fictional scenario in which $w_{19} = w_{20}$, 24 holds.

(24) $[w_{19}/(w_{19} + k)] = [w_{20}/(w_{20} + k)] = c_x = c_y$

Still, all that has been said leads to (25).

(25) $[(w_{19}^{+} + k/2)/(w_{19} + k)] < [(w_{20}^{+} + k/2)/(w_{20} + k)]$

Although $w_{19} = w_{20}$, given that $w_{19}^{+} < w_{20}^{+}$, we must admit that $f_x < f_y$. Accordingly,

(26) $[c_x \times (f_x - ½) + ½] < [c_y \times (f_y - ½) + ½]$

Let $e(19)$ and $e(20)$ be the expectation values of, respectively, (19) and (20). If (25) and (26) are the case, then (27) is the case.

(27) $e(19) < e(20)$

But (27) leads to respond to (21) with *Possible*. As indicated, depending on the data the system has, the answer can be a term different from both *Possible* and *Impossible*. However, what appears to be undeniable is that *Possible* is always preferable over *Impossible* as a response to (21).

Conclusions

Hintikka's theorem has been demonstrated following general technical requirements of classical, modal, and deontic logics. In the literature, we can find explanations based on psychological theories accounting for why, despite that, people can tend not to accept the theorem.

In this paper, I have tried to do the same within NAL framework. What the theorem provides can be expressed as an inheritance statement in NAL. Quantifiers are not necessary in the latter logic; it includes truth values such as f and c that seem to eliminate their necessity. On the other hand, the isomorphism between the material conditional in classical logic and the copula in inheritance statements in NAL also helps convert what the theorem indicates into an inheritance statement. In addition, the operators of possibility and permission, from, respectively, modal logic and deontic logic, can be understood as terms in NAL.

From this point on, we can calculate the expectation value for both a statement indicating that what is impossible is forbidden and a statement establishing that what is possible is forbidden. Given that it is evident that the second statement will have more positive evidence than the first one, if the two statements have the same confidence value, the second statement will have a higher expectation value.

By virtue of the choice rule, the higher expectation value means that the statement linking what is possible to what is forbidden should be selected before the statement relating what is impossible to what is forbidden. Therefore, in NAL, if we ask about the subject of the predicate *Forbidden*, the tendency will be to prioritize *Possible*. One might think that this is more like the way people can understand the theorem.

NAL has much more resources and components than those described in the present paper. There are other manners to address Hintikka's theorem from NAL. Those manners might be different in terms of simplicity and rigor from mine here. However, they can hardly lead to opposite conclusions. It is difficult to accept Hintikka's theorem in NAL.

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